SPECIAL ISSUE

INSIGHTS INTO THE PHYSICS AND THE SOLUTIONS OF FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS

GUEST EDITORS

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PREFACE

Motivation

This work was originally presented as a series of NASA reports between 1998 and 2000. The next four sections will provide an overview of each of the four papers, as well as some insight into the importance of the problems addressed. Several new books on the general topic of the fractional calculus have been written in the interim [1]-[3] and some updated information is also provided.

Paper 1: Insights into the Fractional Order Initial Value Problem via Semi-infinite Systems

One of the first issues that many newcomers ask when beginning their study of fractional calculus, or fractional-order systems, is whether anything in the real world behaves as if it is of fractional order. Stated otherwise that question is, "Does anything behave that way?" or "Is there any fundamental fractional-order physics?" The first paper in this issue attempts to answer these questions in an affirmative way by demonstrating that the physics described by the diffusion equation can have an impedance that is of half-order. The diffusion equation describes the physics of many systems including the diffusion of a material into a medium, the conduction of heat through a thermal medium, the percolation of a liquid through a porous medium, the displacement of a viscoelastic material under load, and as shown in the first paper, the transmission of an electrical signal into a lossy line. When the spatial dimension is "large," the impedance becomes a semi-integral for such systems. By impedance, we mean the ratio of the across variable to the through variable, that is voltage to current, or temperature to heat flow, for example. Thus, for long lines, the voltage at the terminal is the semi-integral of the current into the terminal, or equivalently, the temperature at the end of a long metal bar is equal to the semi-integral of the heat flowing into the bar.

The other main purpose of this first paper is to provide a clear inference that fractional-order systems must have time-varying initialization functions to be useful in the description of fundamental physics. The initialization function maps the historical behavior of the given system into a future response. Thus for the semi-infinite line, the voltage at the terminal of the line is the semi-integral of the current into the line as mentioned above, plus the initialization function. That is,

$$
\begin{aligned}
v(t) &= {}_0D_t^{-1/2} i(t) = {}_0d_t^{-1/2} i(t) + \psi(t) \\
&= \frac{1}{\Gamma(1/2)} \int_0^t (t-\tau)^{-1/2}\, i(\tau)\, d\tau \;+\; \frac{1}{\Gamma(1/2)} \int_a^0 (t-\tau)^{-1/2}\, i(\tau)\, d\tau
\end{aligned}
$$

where v is voltage, i is current, and t is time, and a is the value of negative time at which the system is turned on, and the history information is contained in the second integral. In the Laplace domain this can be written as

$$V(s) = \frac{1}{s^{1/2}} I(s) + \psi(s)\,.$$

This last equation comes directly from the solution of the semi-infinite lossy line problem as shown in the first paper, and provides a clear physical basis for the need of a time-varying initialization function in fractional calculus. Earlier Laplace transforms for fractional derivatives, such as the Laplace transform of the Caputo derivative, did not properly take into account the requirement for a time-varying initialization.

To conclude this discussion, this paper shows that there are physical systems that display fractional-order behavior, at least over broad frequency ranges. Without fractional calculus, such systems often required complicated mathematical representations. The fractional calculus provides a simple and compact description of such systems.

Paper 2: A Solution to the Fundamental Linear Fractional Order Differential Equation

Once it is accepted that fractional-order systems exist in nature, the difficulty then becomes one of finding the mathematical solution to physical problems containing such systems. For dynamic problems, differential equations are usually the starting point for this analysis. With fractional-order building blocks, fractional-order differential equations result. The simplest possible fractional differential equation is presented and analyzed in the second paper in this issue,

$${}_0d_t^q x(t) + a\, x(t) = u(t), \qquad q > 0,\ t > 0,$$

where x is a system dynamic variable, u is an input to the system, and a is a constant characteristic of the given system. This equation leads to the transfer function

$$\frac{X(s)}{U(s)} = \frac{1}{s^q + a}\,.$$

Most time-domain analysis of such a transfer function requires the knowledge of the system impulse response, which is the inverse Laplace transform of the transfer function. In the integer-order calculus, this function would be the exponential function. The second paper derives this fractional-order impulse response and names it the *F*-function,

$$L\left\{F_q(-a,t)\right\} = \frac{1}{s^q + a}, \quad q > 0.$$

The *F*-function should be considered as a generalization of the exponential function.

The paper then shows that the transfer function can be analyzed in the Laplace plane, but the presence of the fractional power of the Laplace variable requires the use of multiple Riemann-sheets in the placement of the system poles and zeros. The Laplace plane analysis is considerably simplified by the use of a new complex variable *w*, defined as $w = s^q$. The behavior of large complicated transfer functions containing commensurate-order fractional terms, can then be readily analyzed in the complex *w*-plane. An example is included to demonstrate solution of a larger transfer function using partial fraction expansions in the *w*-variable. A recent study by Malti [4] on transfer functions with three commensurate denominator terms extends this work.

Paper 3: Generalized Functions for the Fractional Calculus

Fractional-order systems have been studied for many years, and many authors have attempted to solve fractional-order differential equations. Consequently, over the years, a plethora of functions along with their Laplace transforms have been presented to aid in the solution of fractional-order differential equations. Many of these functions are presented in the third paper in this issue.

Many fractional generalizations of the exponential function exist that result in the common exponential function when the order-term goes to unity. The authors, of course, think the *F*-function is the most appropriate function to use, as it is the impulse response of the fundamental system. Other less directly applicable functions are now discussed. An often used function is a modification of the Mittag-Leffler function as is used by Podlubny [1]. Miller and Ross [5] present a function that is a series of the fractional derivatives of the exponential function. Glockle and Nonnenmacher [6] present functions in terms of the even more complicated Fox Functions. Robotnov [7] studied a function related to hereditary integrals, and created an extensive set of tables [8]. This function is not the *F*-function as stated in the third paper, but related by a power of the Laplace *s*-variable. Padovan and Sawicki [9] also discuss a similar function in the context of viscoelasticity and constitutive equations. Even more recently, Matignon [10] derives some stability properties for these functions.

This third paper presents another function of considerable utility in dealing with fractional-order systems, the *R*-function,

$$R_{q,v}[a,t] \equiv \sum_{n=0}^{\infty} \frac{a^n \, t^{(n+1)q-1-v}}{\Gamma((n+1)q-v)} \Leftrightarrow \frac{s^v}{s^q - a}, \qquad q-v>0.$$

This function generalizes the *F*-function, and is useful because the fractional derivatives and integrals of the *F*-function are built directly into it. The *R*-function is similar to Podlubny's two-variable Mittag-Leffler function

$$\xi(t,a,q,v) = \sum_{n=0}^{\infty} \frac{a^n t^{qn+v-1}}{\Gamma(nq+v)} \Leftrightarrow \frac{s^{q-v}}{s^q - a}$$

but easier to use in practice as it is the fractional derivative or integral of the impulse response. The paper shows that many of the higher functions in common use can easily be written in terms of the *R*-function, as well as most of the fractional generalizations of the exponential function. Additionally, some interrelationships among *R*-functions are presented.

Paper 4: R-Function Relationships for Application in the Fractional Calculus

The final paper recognizes the importance and pervasiveness of the *R*-function, and consequently pursues interrelationships among *R*-functions and the common exponential function. This paper, along with the third paper, provides expressions for the elementary functions, as well as some advanced functions in terms of the *R*-function. Properties of the *R*-function, and many *R*-function based identities are derived. Plots of the *R*-functions are given. Extensive relationships between the *R*-functions of various commensurate orders (q) are derived and presented. Relationships between the *R*-function and the exponential function are also derived. The paper concludes by giving several rational approximations to the *R*-function.

Summary

The authors have three goals for these papers. One goal is to make the introduction to the fractional calculus easier for the reader than it was for the authors. The second goal is to further reinforce the necessity of the time-varying initialization function for any fractional-order operator in the solution of fractional differential equations. Finally, the third goal is encourage the use of *F*-functions and *R*-functions in the solution of fractional differential equations.

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